Parameter-free Online Optimization
Part 3

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Outline of the Tutorial

- Part 1: Stochastic and Online Convex Optimization
- Part 2: Parameter-free Convex Optimization
- **Part 3: More Adaptivity and Applications**
- Part 4: Implementation, Experiments, Open Problems

Previously

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- We saw that stochastic optimization can be solved via online linear optimization:

$$\mathbb{E}\left[\sum_{t=1}^{T} F(\boldsymbol{x}_t) - F(\boldsymbol{x}^\star)\right] \le \mathbb{E}\left[\sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{x}_t - \boldsymbol{x}^\star \rangle\right]$$

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- What next?

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Building algorithms that go beyond worst-case tuning.

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   - We will assume $\boldsymbol{x}_1 = 0$ in this section to avoid writing $\boldsymbol{x}^\star - \boldsymbol{x}_1$ too much.

Second-Order Regret Bound

We will build an online linear optimization algorithm that obtains:

$$\sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{x}_t - \boldsymbol{x}^\star \rangle \le \tilde{O}\left(\|\boldsymbol{x}^\star\| \sqrt{\sum_{t=1}^{T} \|\boldsymbol{g}_t\|^2} \right)$$

This is the “AdaGrad version” of the KT-estimator regret bound $\tilde{O}\left(\|\boldsymbol{x}^\star\| G\sqrt{T} \right)$.

[Cutkosky&Orabona, COLT’18; Kempka et al., ICML’19; Van der Hoeven, NeurIPS’19; Mhammedi&Koolen, COLT’20]

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Let us see how to obtain this bound. For simplicity, we will assume $G = 1$.

[Cutkosky&Orabona, COLT'18; Kempka et al., ICML'19; Van der Hoeven, NeurIPS'19; Mhammedi&Koolen, COLT'20]

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- This is actually no restriction: any algorithm that guarantees $Regret_T(0) = O(1)$ must be a coin-betting algorithm.

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2. Observe that this fixed $\beta^\star$ yields the second-order bound.
3. Find a way to generate β_t on-the-fly that does nearly as well as $\beta^\star$.

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Whenever you see a product, it's natural to take the logarithm:

$$\log(\text{Wealth}_T) = \sum_{t=1}^{T} \log(1 - \beta_t g_t)$$

A Good Fixed Betting Fraction

Let us guess $\beta^\star = -\frac{\sum_{t=1}^T g_t}{\sum_{t=1}^T g_t^2 + 2\left|\sum_{t=1}^T g_t\right|}$. This yields $\beta^\star \in [-1/2, 1/2]$, and so

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The Story So Far

1. There is a fixed betting fraction $\beta^\star$ that would guarantee a second-order regret bound.
2. How can we learn about $\beta^\star$ on-the-fly?

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[Cutkosky&Orabona, COLT'18]

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$$:= \sum_{t=1}^{T} \ell_t(\beta_t) - \sum_{t=1}^{T} \ell_t(\beta^\star)$$

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ℓ_t is exp-concave! This means that choosing β_t by gradient descent on ℓ_t (e.g. via Online Newton Step [Hazan et al., MLJ'07]) yields:

$$\leq \log(T)$$

$$\text{Wealth}_T \geq \frac{\text{Wealth}^\star_T}{T}$$

[Cutkosky&Orabona, COLT'18]

Wrapping Up Second Order Bound

Now we have

$$\begin{aligned}\text{Wealth}_T &\geq \frac{\text{Wealth}_T^\star}{T} \\ &\geq \frac{1}{T}\exp\left[\frac{\left(\sum_{t=1}^T g_t\right)^2}{2\sum_{t=1}^T g_t^2 + 4\left|\sum_{t=1}^T g_t\right|}\right] := H\left(\sum_{t=1}^T g_t\right)\end{aligned}$$

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The $\frac{1}{T}$ only changes a $\log(1+|x^\star|\sqrt{T})$ to $\log(1+|x^\star|T)$.

[Cutkosky&Orabona, COLT'18]

Full Second-Order Bound Algorithm

$\beta_1 = 0, \ \text{Wealth}_0 = 1$
for $t = 1$ **to** T **do**
  Play $x_t = \beta_t \text{Wealth}_{t-1}$
  Get gradient g_t, define $\ell_t(\beta) = -\log(1 - \beta g_t)$
  Compute $z_t = \ell_t'(\beta_t) = \frac{g_t}{1-\beta g_t}$
  Set $\beta_{t+1} = \text{clip}\left(\beta_t - \frac{z_t}{1+\sum_{i=1}^t z_i^2}, -\frac{1}{2}, \frac{1}{2}\right)$
  This is the ONS update
  Set $\text{Wealth}_t = \text{Wealth}_{t-1} - g_t x_t$
end for

Theorem

The above algorithm guarantees

$$\sum_{t=1}^{T} g_t(x_t - x^\star) \leq \tilde{O}\left(|x^\star| \sqrt{\sum_{t=1}^{T} g_t^2 \log(|x^\star| T)}\right)$$

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1. A bound that depends on the sparsity of $\boldsymbol{x}^\star$.
2. A bound that automatically adapts to the smoothness and variance parameters.
3. A bound that automatically adapts to strong-convexity parameters.

Sparsity via Per-Coordinate Updates

Remember one way to make a d-dimensional algorithm from a 1-dimensional algorithm, ala AdaGrad:

$$\sum_{t=1}^{T} \langle g_t, \boldsymbol{x}_t - \boldsymbol{x}^\star \rangle = \sum_{i=1}^{d} \underbrace{\sum_{t=1}^{T} g_{t,i}(x_{t,i} - x_i^\star)}_{\text{Set } \boldsymbol{x}_{t,i} \text{ via a 1-d learner.}}$$

$$\leq \underbrace{\tilde{O}\left(\sum_{i=1}^{d} |x_i^\star| \sqrt{\sum_{t=1}^{T} g_{t,i}^2}\right)}_{\text{The } \tilde{O} \text{ now hides a } \log(d) \text{ factor.}}$$

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Contrast with the (optimally tuned) AdaGrad guarantee:

$$\sum_{t=1}^{T}\langle g_t, \boldsymbol{x}_t - \boldsymbol{x}^\star\rangle \leq O\left(\max_i |x_i^\star| \sum_{i=1}^{d}\sqrt{\sum_{t=1}^{T} g_{t,i}^2}\right)$$

Sparsity via Per-Coordinate Updates

What happens if some coordinates $x_i^\star$ are 0?

- Suppose $\|\boldsymbol{x}^\star\|_\infty \leq 1$, $\|\boldsymbol{g}_t\|_\infty \leq 1$, and $\boldsymbol{x}^\star$ is S-sparse. Then:

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&\le \tilde{O}\left(\|\boldsymbol{x}^\star\|_1 \sqrt{\sum_{t=1}^{T} \|\boldsymbol{g}_t\|_\infty^2} \right) \\
&\le S\sqrt{T}
\end{aligned}$$

More Properties of the Per-Coordinate Update

Per-Coordinate updates also achieve the L_2-norm bound:

$$\sum_{t=1}^{T}\langle g_t, \boldsymbol{x}_t - \boldsymbol{x}^\star\rangle \le \tilde{O}\left(\sum_{i=1}^{d}|x_i^\star|\sqrt{\sum_{t=1}^{T} g_{t,i}^2}\right)$$

$$\text{apply Cauchy-Schwarz} \le \tilde{O}\left(\|\boldsymbol{x}^\star\|\sqrt{\sum_{t=1}^{T}\|\boldsymbol{g}_t\|^2}\right)$$

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⚠ This is not the same as identifying which coordinates of $\boldsymbol{x}^\star$ are zero.

[Kotłowski, ALT'17]

Parameter-Free Experts Algorithms

- Learning with expert advice setting is a special case of online linear optimization in which both $\boldsymbol{x}_t$ and $\boldsymbol{x}^\star$ must lie in the probability simplex in $\mathbb{R}^d$.
- There are many parameter-free algorithms for this special case, starting with NormalHedge [Chaudhuri et al., NeurIPS'09], and including [Gaillard et al., COLT'14; Koolen&van Erven, COLT'15; Luo&Schapire, COLT'15; Foster et al., NeurIPS'15; Harvey et al., arXiv'20].
- In this special setting, one can hope for interesting different bounds. For example, the Squint algorithm can obtain:

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Parameter-Free Experts Algorithms

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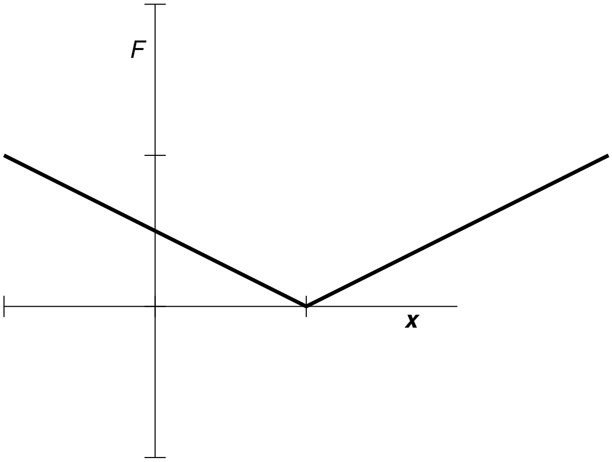
- This does not have the $g_{t,i}^2$, but works in the simplex.

Adapting to Smoothness



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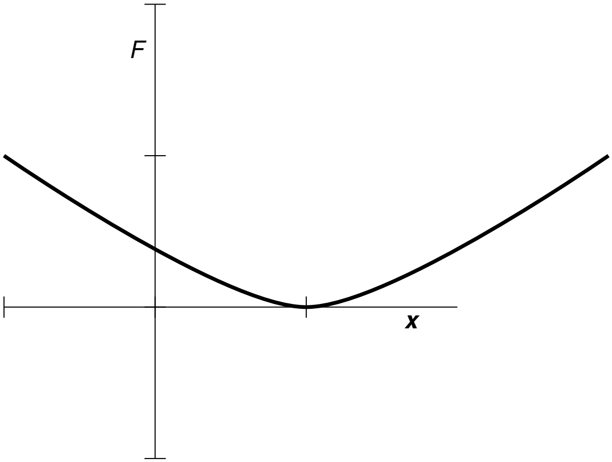

Adapting to Smoothness

Suppose our original objective $F(\boldsymbol{x})$ is L-smooth:

$$\nabla^2 F(\boldsymbol{x}) \preceq LI$$

We don't know what L is. Suppose also that for all t,

$$\mathrm{Var}[\boldsymbol{g}_t] \leq \sigma^2$$

In this case, the optimal learning rate for gradient descent depends on L and σ, and achieves:

$$\mathbb{E}[F(\boldsymbol{x}_{\mathrm{SGD}}) - F(\boldsymbol{x}^\star)] \leq \frac{L\|\boldsymbol{x}^\star\|^2}{T} + \frac{\sigma\|\boldsymbol{x}^\star\|}{\sqrt{T}}$$

[Ghadimi&Lan, J. of Opt.'12]

Adapting to Smoothness

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- To see how, we need to look inside the online-to-batch conversion, and look beyond the linear approximation to F.
- Specifically, we need the following simple consequence of smoothness (see [Srebro et al., NeurIPS'10; Levy, NeurIPS'17] for related applications):

$$\mathbb{E}[\|\boldsymbol{g}_t\|^2] \le L(F(\boldsymbol{x}_t) - F(\boldsymbol{x}^\star)) + \sigma^2$$

Adapting to Smoothness

In the online-to-batch conversion, we argued:

$$\mathbb{E}\left[\sum_{t=1}^{T} F(\boldsymbol{x}_t) - F(\boldsymbol{x}^\star)\right] \leq \mathbb{E}\left[\sum_{t=1}^{T}\langle \boldsymbol{g}_t, \boldsymbol{x}_t - \boldsymbol{x}^\star\rangle\right] \text{ (convexity)}$$

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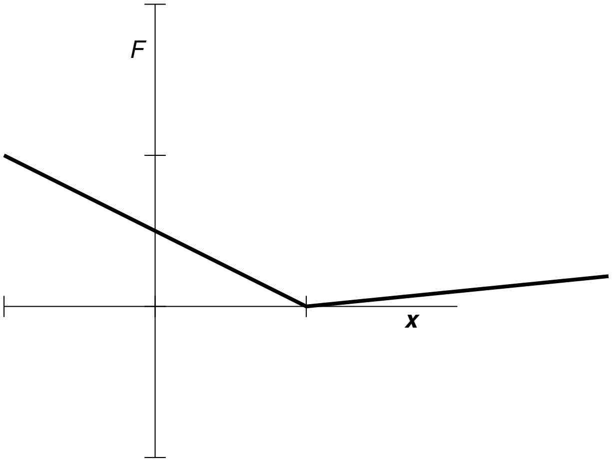
Adapting to Strong Convexity



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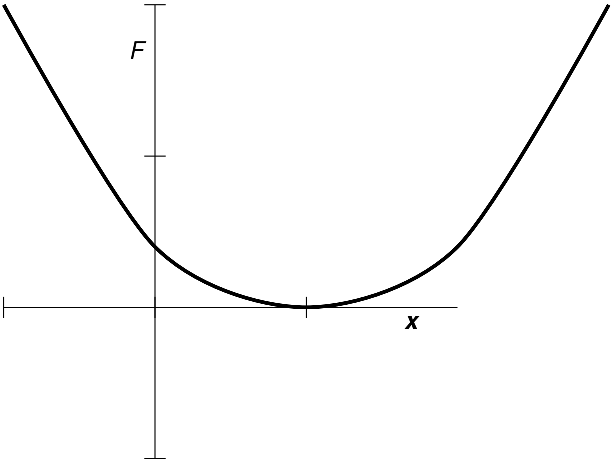

Adapting to Strong-Convexity

- The strongly-convex losses are an important special class of optimization problems:

$$\nabla^2 F(\boldsymbol{x}) \succeq \mu I$$

- For strongly-convex losses, gradient descent with learning rate $\frac{1}{\mu t}$ guarantees [Hazan et al., MLJ'07]:

$$\mathbb{E}[F(\boldsymbol{x}_{\text{SGD}}) - F(\boldsymbol{x}^\star)] \leq \frac{\log(T)}{\mu T}$$

This requires knowledge of the parameter μ. Can we do without this?

A Regret Bound for Strong-Convexity

- MetaGrad [Koolan&van Erven, NeurIPS'16] and Maler [Wang et al., UAI'19] are online linear optimization algorithms that guarantee:

$$\sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{x}_t - \boldsymbol{x}^\star \rangle \leq \tilde{O}\left(\sqrt{\sum_{t=1}^{T} \|\boldsymbol{x}_t - \boldsymbol{x}^\star\|^2 \|\boldsymbol{g}_t\|^2}\right)$$

- Amazingly, this automatically implies adaptivity to strong-convexity!
- We'll look at a simple trick for obtaining this style of bound.

Dynamics of $\boldsymbol{x}_t$

Recall:

- We obtain error $\tilde{O}\left(\frac{\|\boldsymbol{x}^\star\| G}{\sqrt{T}}\right)$ (ignoring logs and constants).
- Intuitively, when $\boldsymbol{x}^\star$ is far from the starting point, we have more space to explore, so we expect to $\boldsymbol{x}_t$ to move a lot..

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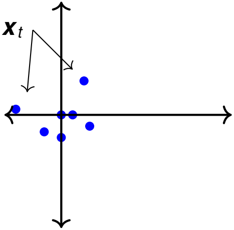
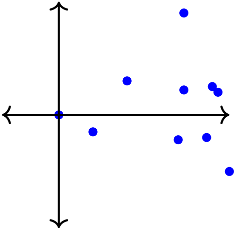


$\boldsymbol{x}^\star = (0,0)$



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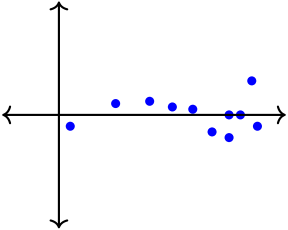
- Get much “tighter” distribution of $\boldsymbol{x}_t$s when $\boldsymbol{x}^\star$ is small.

Dynamics of $\boldsymbol{x}_t$

Can we have a tight distribution even when $\boldsymbol{x}^\star$ is not small?



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- We can get suboptimality

$$\mathbb{E}\left[F\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_t\right) - F(\boldsymbol{x}^\star)\right] = \tilde{O}\left(\frac{\|\boldsymbol{x}^\star\| G}{\sqrt{T}}\right)$$

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- If we shift all outputs $\boldsymbol{x}_t$ by a constant $\boldsymbol{v}$, then we get

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- If $\boldsymbol{v}$ is close to $\boldsymbol{x}^\star$, then this is much better.
- So how can we find a closer $\boldsymbol{v}$?

Using Strong Convexity

We need to move beyond the naive linear approximation $F(\boldsymbol{x}_t) - F(\boldsymbol{x}^\star) \le \langle \boldsymbol{g}_t, \boldsymbol{x}_t - \boldsymbol{x}^\star \rangle$. Strongly-convex functions also satisfy:

$$F(\boldsymbol{x}) - F(\boldsymbol{x}^\star) \ge \frac{\mu}{2} \|\boldsymbol{x} - \boldsymbol{x}^\star\|^2$$

This means that <u>points with low error are close to $\boldsymbol{x}^\star$</u>.

Adapting to Strong Convexity

- Small function value means that we are close to $\boldsymbol{x}^\star$.

[Cutkosky&Boahen, NeurIPS'17; Cutkosky&Orabona, COLT'18]

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- Via some careful algebra, we get:

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  This implies $\tilde{O}\left(\frac{1}{\mu T}\right)$ error in strongly-convex problems.
- Adding the average works for <u>any</u> parameter-free algorithm.

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- This pays linearly for any over-estimate in B.

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  - Note that the extra $\|\boldsymbol{x}^\star\|^3$ term <u>does not depend on T</u>.
  - It is unclear how "real" this lower bound is in real stochastic environments.
- The bound is tight. It can be obtained "lying" to the algorithm by feeding $\boldsymbol{g}_t$ clipped to have norm at most $\max_{i<t} \|\boldsymbol{g}_i\|$ [Cutkosky, COLT'19; Mhammedi&Koolen, COLT'20].

Summary of Part 3

Once you know how to make parameter-free algorithms, it is surprisingly easy to build extra results.

1. Adapt to sparse $\boldsymbol{x}^\star$.
2. Adapt to smoothness.
3. Adapt to strong-convexity.
4. Incorporate unknown G.